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The Thermal Response of a Metal
Slab to a Class of Radially
Dependent Heat Inputs

David C. Stickler
Antenna Laboratory
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Report 1107-14

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THE THERMAL RESPONSE OF A METAL
SLAB TO A CLASS OF RADIALLY
DEPENDENT HEAT INPUTS

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Prepared for:

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ABSTRACT

The temperature fields are deduced for an infinite metal slab of finite thickness produced by a heat flux input which depends on the radial coordinate. The formal solution is obtained by means of transform techniques, and several alternative approaches to their evaluation are presented. Two heat flux inputs are considered in detail. They are the Gaussian input defined in Eq. (E-1), and the uniform input over a circular area of the front surface. Such inputs might be used, for example, as models for a study of the thermal-elastic effects of a laser beam focused on a metal slab.

Several asymptotic expansion are presented for the small-time response of the temperature fields under various assumptions on the heat flux input.

An extension of the results of Oosterkamp[3] has been derived as well as a verification of the validity of the approximations through the derivation of an upper bound on the error.

PUBLICATION REVIEW

This report has been reviewed and is approved.

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DEPENDENT HEAT INPUTS

A. Introduction

The temperature fields produced by a heat flux input dependent
on the radial coordinate only are deduced for an infinite metal slab of
finite thickness.

The formal solution is obtained by the use of transform techniques,
and several alternative approaches to their evaluation are presented.

Attention is focused on a class of input flux densities which are
concentrated at the origin, such as would occur if a laser beam were
focused to a small spot on one surface of the slab. Two examples of
this type which have been considered in detail are the Gaussian and
uniform inputs defined in Eqs. (E-1) and (F-1), respectively. Unfortu-
nately, extensive numerical information can be obtained only at the cen-
ter of the front surface. In terms of the geometry described in Fig. 1,
this is at $\rho = \xi = 0$. This is, however, a very interesting point since
the temperature maximum for the class of concentrated inputs occurs
here. The temperatures at this point also illustrate some of the dis-
tinctive features of the problem.

One of the basic results shows that for time sufficiently small,
the response is essentially a local phenomenon. Specifically, this means
that if $q(\rho)$ is a sufficiently smooth function of ρ, the primary diffusion
is into the slab while the effect of radial diffusion is secondary for time
small. This in turn means that the temperature response under these
conditions should be given to a good approximation by the product of the
local input multiplied by the temperature response for the flat plate with
a uniform input on the front surface. In this paper several results of
this type will be given, each with a different type of "smoothness"
requirement. Remainders will indicate their validity.

The paper can be divided into two parts. The first part deals
with the general solution to the problem, and its various formulations.
The second part deals with two specific inputs for which more explicit
results can be obtained. These results are presented in analytical and
graphical form when possible.
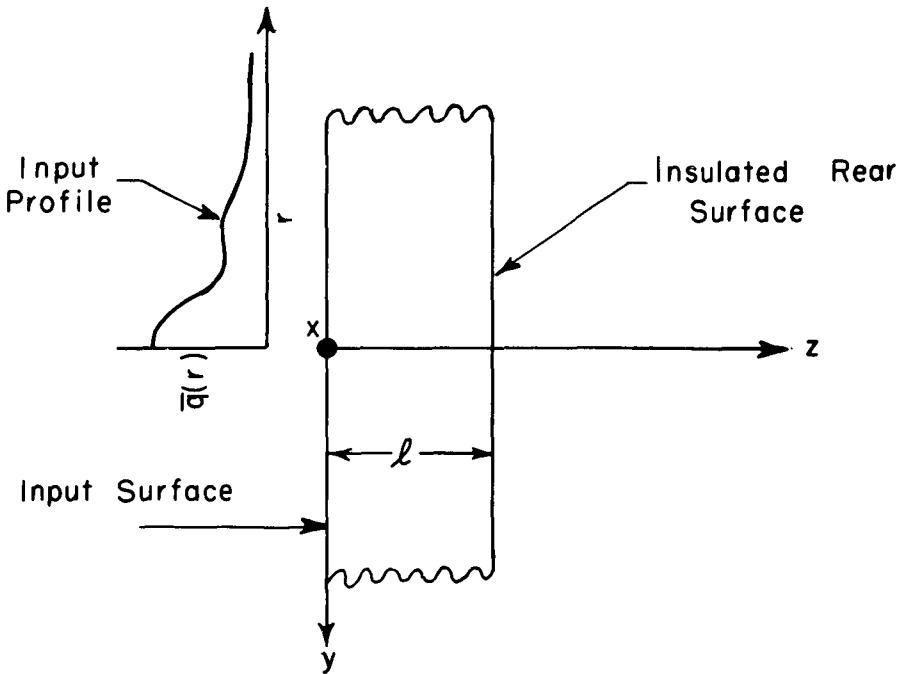
Fig. 1. This figure illustrates the geometry of the problem.
The heat flux input occurs on the front surface
(Z = 0) and depends on the radial coordinate only.
The rear surface (Z = ℓ) is insulated.

This problem has, of course, been studied by others, but the
main results of the other authors have been for the semi-infinite body
with a uniform input over a portion of the surface and for final temper-
atures in the cases where a linear radiation law is included.

Carslaw and Jaeger[1] give several references to this type of
problem, and other than these references it has not been possible to
find any others. Lowan[2] considers the semi-infinite body with a more
general time dependence than that considered here. In the present
report the input flux has a step dependence with respect to time and is
a basis, through Duhamel's theorem, for the study of the more general
time dependence problem.

Oosterkamp[3] considers a uniform spot input on the front surface of a semi-infinite body. He reasoned correctly that the temperature response for the semi-infinite body would furnish first-order approximation for the finite slab for time small. This is because for time small the presence of the rear wall is not significant, nor is the effect of radial diffusion. He did not present, however, an evaluation of this approximation. This paper extends his result and verifies his reasoning by giving a rigorous estimate of the error. While the approximation itself does not depend on the thickness of the slab, the error estimate appropriately does.

Thomas[4] considers the semi-infinite body and the very thin slab. His results for the semi-infinite body include a linear radiation loss. The input to his thin-slab problem is not a surface flux but a constant volume input restricted to a right-circular cylinder domain normal to the slab surface.

The motivation for the present problem is similar to Oosterkamp's. He was interested in the thermal effects of electron bombardment of the anode of an X-ray tube, and how to prolong the life of the tube. In the present problem the interest is in determining the effect of the focusing of a laser[5] output on a metal slab. The laser is a device which radiates an almost coherent electromagnetic energy at optical frequencies. This energy from the laser can be focused to a very small spot. A typical diameter is one millimeter. The resulting incident power densities are about 10^{10} watts/meter^2. The coupling of electromagnetic energy to metals is quite efficient at optical frequencies, especially when compared to the coupling at microwave frequencies. In some cases the laser beam will punch a small hole in a thin metal slab. It is felt that the first step in a study to determine the cause of this punching is an analysis of the temperature fields. From this study, then, the study of the thermally induced stress fields can be initiated.[6]

Two recent papers[30,31] indicate that the solution to this problem can also be of value in improving the accuracy of the determination of the thermal parameters. First, the effect of the shape of the beam can be taken into account and secondly the plate size can be made such that the edges of the plate don't effect the determination.

B. Formal Solution

   In this section the formal solution to the problem will be presented in two alternate forms: a Green's function form, and a transformed Green's function form. Both are obtained by transformation techniques which are quite natural for the problem. The Laplace transform is used because of the time dependence of the input flux density; it is zero for time negative. The Hankel transform of zero order on the radial dependence is used because of the cylindrical symmetry of the input flux density. These two transformations reduce the problem to that of finding the solution for the plane slab problem with a uniform input over the entire surface and performing two inverse transforms. The use of one or the other of the two forms of the solution depends on the input flux density $q(\rho)$ and its Hankel transform, $Q(\lambda)$. The geometry of this problem is shown in Fig. 1 and the differential equations and boundary conditions are given in Eq. (B-1):

(B-1)
$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\,\frac{\partial \Delta U}{\partial \rho}\right) + \frac{\partial^2 \Delta U}{\partial \xi^2} - \frac{\partial \Delta U}{\partial \tau} = 0, \begin{cases} 0 \le \rho < \infty \\ 0 \le \xi \le 1 \\ 0 \le \tau \end{cases}$$

$$\frac{\partial \Delta U(\rho,0,\tau)}{\partial \xi} = -q(\rho); \quad 0 \le \rho < \infty, \qquad \tau > 0$$

$$= 0 \quad ; \qquad \qquad \tau < 0$$

$$\frac{\partial \Delta U(\rho,1,\tau)}{\partial \xi} = 0 \qquad 0 \le \rho, \tau < \infty$$

$$\Delta U(\rho,\xi,0) = 0 \qquad \begin{matrix} 0 \le \rho < \infty \\[4pt] 0 \le \xi \le 1 \end{matrix}$$

where $\Delta U(\rho,\xi,\tau) = T(r,z,t) - T_O$ and $\xi = z/\ell$, $\rho = r/\ell$; ℓ = slab thickness; $\tau = Kt/\ell^2$; K is thermal diffusivity, t is time; $q(\rho) = \overline{q}(r)\ell/k$; k is thermal conductivity.

   The normalized heat flux input density $q(\rho)$ will be called the heat flux input density or heat flux input. The units of the normalized heat flux input density $q(\rho)$ in the MKS system are in degrees Kelvin. Later in the paper the symbol Q with no argument will be used to denote the total heat input. In this case it will be given by

$$Q = 2\pi \int_{\rho=0}^{\infty} \bar{q}(r) \; r$$

and measured in watts; that is, Q is the actual total heat input.

The formal solution using the Laplace transform and Hankel transform techniques is given by the expression

(B-2)
$$\Delta U(\rho, \xi, \tau) = \frac{1}{2\pi i} \int_{s=0_+ - i\infty}^{0_+ + i\infty} \int_{\lambda=0}^{\infty} \frac{Q(\lambda)}{s}$$

$$\left\{ \frac{\cosh\{\sqrt{\lambda^2 + s}\;(1-\xi)\}}{\sqrt{\lambda^2 + s}\;\sinh\sqrt{\lambda^2 + s}} \right\} \cdot e^{s\tau} \cdot J_0(\lambda\rho)\lambda$$

where $Q(\lambda)$ is the Hankel transform of the input flux density, $q(\rho)$, and

(B-3)
$$Q(\lambda) = \int_{\rho'=0}^{\infty} q(\rho') \; J_0(\lambda\rho') \; \rho'$$

From this point there are indeed many paths to follow. As can be seen, there are three integrations to perform and thus there are a multitude of possible approximations which can be made. It is impossible to discuss them all. Therefore, the remainder of this report discusses the methods most advantageous to the needs of the laser problem mentioned earlier.

First, the integration with respect to s will be carried out. There seems to be no alternative other than this. Notice that if λ were zero the bracketed term given below (from the integrand in Eq. (B-2)),

(B-4)
$$H(\xi, \lambda, s) = \frac{1}{s} \frac{\cosh\{\sqrt{\lambda^2 + s}\;(1 - \xi)\}}{\sqrt{\lambda^2 + s}\;\sinh\sqrt{\lambda^2 + s}} \quad,$$

would be the Laplace transform of the slab problem for a uniform heat flux input over the entire front surface.[7] In terms of the present problem the input flux $q(\rho)$ would be a constant for all ρ and although $Q(\lambda)$ would not exist in the ordinary sense, it would be a functional, namely, $\delta(\lambda)$. When this is substituted in Eq. (B-2) the result is the constant flux solution. This is simply a statement of Hankel's integral formula.[8] For $\lambda \neq 0$ there is an interesting relationship between this problem and the thin-wire problem with linear radiation from the lateral surface as discussed by Sommerfeld.[9] Basically, the two problems are quite similar. This fact is apparent when one notes that the differential equation satisfied by the inverse Laplace transform of $H(\xi, \lambda, s)$ is the same as the thin-wire problem heated uniformly at one end. The variable λ corresponds to the radiation loss parameter of Sommerfeld's problem. The solution to the present problem is then seen as an integration over all possible radiation loss parameters multiplied by an appropriate weight factor. The main difference is that in the present problem the radial diffusion is caused by the unequal heat flux input on the front surface rather than a radiation loss from the lateral surface. Also when no radiation loss is present ($\lambda = 0$) one expects to obtain the flat-plate solution with no radiation loss. This is what is obtained, as mentioned earlier.

Performing the inverse Laplace transform one arrives at the following expression for the temperature field; this will be called the transformed Green's function representations.

(B-5) $$\Delta U(\rho, \xi, \tau) = \int_{\lambda=0}^{\infty} Q(\lambda) \, F(\rho, \xi, \tau; \lambda) \, \lambda$$

where $$F(\rho, \xi, \tau; \lambda) = J_0(\lambda \rho) \int_{\tau'=0}^{\tau} \theta_3 \left(\frac{\xi}{2}, i\pi\tau' \right) e^{-\lambda^2 \tau'}$$

and $\theta_3(\xi/2, i\pi\tau')$ is a theta function.[10] The two representations of the theta function are given in Eqs. (B6a) and (B-6b).

(B-6a) $$\theta_3 \left(\frac{\xi}{2}, i\pi\tau \right) = 1 + 2 \sum_{n=1}^{\infty} \cos(n\pi\xi) e^{-n^2 \pi^2 \tau}$$

(B-6b) $\quad \theta_3 \left(\dfrac{\xi}{2} , i\pi\tau \right) = \dfrac{1}{\sqrt{\pi\tau}} \, e^{-\frac{\xi^2}{4\tau}} \, \theta_3 \left(\dfrac{\xi}{2i\pi\tau} , \dfrac{i}{\pi\tau'} \right) \quad .$

Again it is seen that for $\lambda = 0$ the τ' integration yields the temperature fields for the constant heat flux input over the entire front surface.

For $\lambda \neq 0$ the integration with respect to τ' can be performed, but it is not particularly advantageous to do so at present. If the integration is carried out, one of the resulting series is independent of τ. It can be summed, [11] but because of its complexity it is not particularly useful.

Notice that in Eq. (B-5) there are still three integrations to perform. The improvement of Eq. (B-5) over Eq. (B-2) is that the theta function has two representations. One of these converges well for τ small, Eq. (B-6b), and the other for τ large, Eq. (B-6a).

Equation (B-5) can be rewritten in a form calling direct attention to the fact that there are three integrations to perform. This form displays the Green's function for the problem, and for some purposes it is more useful than that given in Eq. (B-5). This form is given by

(B-7) $\quad \Delta U(\rho , \xi, \tau) = \displaystyle\int_{\rho=0}^{\infty} q(\rho) \, G(\rho , \rho' , \xi, \tau) \, \rho'$

where the Green's function is given by

(B-8) $\quad G(\rho , \rho', \xi, \tau) = \displaystyle\int_{\tau'=0}^{\tau} \int_{\lambda=0}^{\infty} J_0(\lambda\rho) \, J_0(\lambda\rho') \, \theta_3 \left(\dfrac{\xi}{2} , i\pi\tau' \right) e^{-\lambda^2 \tau'} \lambda \quad .$

Thus Eqs.(B-5) and (B-7) give two alternate representations of the solution. The use of one or the other depends on whether the input flux density $q(\rho)$ or its transform $Q(\lambda)$ is more convenient to use.

There are several ways to deal with $G(\rho , \rho', \xi, \tau)$, and apparently only one way to deal with $F(\rho , \xi, \tau ; \lambda)$ without explicit information about $Q(\lambda)$ or $q(\rho)$. Unfortunately not all of the results obtained for the Green's function $G(\rho, \rho', \xi, \tau)$ and its transform $F(\rho, \xi, \tau ; \lambda)$ are useful on the special cases. This is because most of the significant results for the Gaussian and uniform inputs depend explicitly and strongly on the particular form of the input flux density or its transform. For each form

$F(\rho, \xi, \tau; \lambda)$ and $G(\rho, \rho', \xi, \tau)$ it is desirable to have both an expansion which converges well for time small and one which converges well for time large. These results are given for $F(\rho, \xi, \tau; \lambda)$ in Section C and for $G(\rho, \rho', \xi, \tau)$ in Section D.

C. The Transformed Green's Function

Two general results based on Eq. (B-5) are deduced in this section. One is an expansion of $F(\rho, \xi, \tau; \lambda)$ for τ large which is the basis for approximations to $\Delta U(\rho, \xi, \tau)$ for time large. The second is an asymptotic expansion for time small, with some restrictions on the input flux density, $q(\rho)$.

The large-time approximation will be deduced first. The first step is to perform the τ' integration. This yields the following:

$$(C-1) \qquad \Delta U(\rho, \xi, \tau) = \int_{\lambda=0}^{\infty} Q(\lambda) \left[\frac{1-e^{-\lambda^2 \tau'}}{\lambda^2} + 2 \sum_{n=1}^{\infty} \frac{\cos n\pi \xi}{\lambda^2 + n^2 \pi^2} \right] J_0(\lambda \rho) \lambda$$

$$- \int_{\lambda=0}^{\infty} Q(\lambda) \left[2 \sum_{n=1}^{\infty} \frac{\cos n\pi \xi}{\lambda^2 + n^2 \pi^2} \exp{-(\lambda^2 + n^2 \pi^2)\tau} \right] J_0(\lambda \rho) \lambda .$$

The first observation is that there are three integrations left. These are as follows:

$$(i) \qquad \int_{\lambda=0}^{\infty} Q(\lambda) \left[\frac{1-e^{-\lambda^2 \tau}}{\lambda^2} \right] J_0(\lambda \rho) \lambda$$

$$(ii) \qquad \int_{\lambda=0}^{\infty} Q(\lambda) \frac{J_0(\lambda \rho) \lambda}{\lambda^2 + n^2 \pi^2}$$

$$(iii) \qquad \int_{\lambda=0}^{\infty} Q(\lambda) \frac{\exp{(-\lambda^2 \tau)}}{\lambda^2 + n^2 \pi^2} J_0(\lambda \rho) \lambda .$$

To ask that all of these be expressable in a simple fashion for some $Q(\lambda)$ is too much. The second observation is that as τ becomes larger, the last integration becomes small quite rapidly. It is easy to verify that this last integration in Eq. (C-1) is bounded by the quantity

$$(\text{C-2}) \quad R_1 < \frac{Q(o)}{6} \frac{e^{-\pi^2 \tau}}{\tau} , \quad 0 \leq \rho < \infty, \quad 0 \leq \xi \leq 1 .$$

This expression does indeed vanish rapidly with increasing τ, and $Q(o)$ is proportional to the total heat flux input in watts to the slab,

$$Q(o) = \frac{Q}{2\pi\ell k} ,$$

where Q is the total heat flux input. In conclusion, then, the temperature change for large τ is given approximately by the expression

$$(\text{C-3}) \quad \Delta U(\rho, \xi, \tau) = \int_{\lambda=0}^{\infty} Q(\lambda) \left[\frac{1-e^{-\lambda^2 \tau}}{\lambda^2} + 2 \sum_{n=1}^{\infty} \frac{\cos n\pi \xi}{\lambda^2 + n^2 \pi^2} \right] J_0(\lambda\rho)\lambda + R_1 ,$$

where $\quad R_1 < \frac{Q}{\ell k} \frac{e^{-\pi^2 \tau}}{12\pi\tau}$

and where Q is the total heat flux input. This is the "large-time" approximation based on the $F(\beta, \xi, \tau; \lambda)$ representation. It yields a term which depends on τ and a term which depends only on the space coordinates. It is assumed that the remaining integration can be performed in a form suitable for numerical computation.

The first integration (i) presents some difficulties for all values of τ. The problem arises in the treatmeatment of the indeterminate form

$$\frac{1 - e^{-\lambda^2 \tau}}{\lambda^2}$$

at $\lambda = 0$. It happens that in this problem $Q(\lambda)$ is an even function of λ (see Eq. (B-3)), and further $Q(\lambda)$ at $\lambda = 0$ is not zero. Thus if one tries to break this integral up into the sum of two,

$$(C-4) \qquad \int_{\lambda=0}^{\infty} Q(\lambda) \left[\frac{1 - e^{-\lambda^2 \tau}}{\lambda^2} \right] J_0(\lambda \rho) \lambda = \int_{\lambda=0}^{\infty} Q(\lambda) \frac{1}{\lambda^2} J_0(\lambda \rho) \lambda$$

$$- \int_{\lambda=0}^{\infty} Q(\lambda) \frac{e^{-\lambda^2 \tau}}{\lambda^2} J_0(\lambda \rho) \lambda \;,$$

it is seen that individually the integrals do not exist. Thus the only way it seems to be possible to handle this integration is by the device used to obtain this representation,

$$(C-5) \qquad \int_{\lambda=0}^{\infty} Q(\lambda) \left[\frac{1 - e^{-\lambda^2 \tau}}{\lambda^2} \right] J_0(\lambda \rho) \lambda = \int_{\tau'=0}^{\tau} \int_{\lambda=0}^{\infty} Q(\lambda) e^{-\lambda^2 \tau'} J_0(\lambda \rho) \lambda \;.$$

This problem will be met again when the Gaussian input is studied and will be handled in this manner. Later the expression in Eq. (C-3) will be used to deduce some information on the temperature fields for the uniform and Gaussian inputs.

Next an asymptotic expansion for the temperature field, $\Delta U(\rho, \xi, \tau)$, as $\tau \to 0$ will be presented. This expansion is discussed more fully in an earlier report.[12] The sufficient assumptions for the expansion are as follows:

$$(C-6) \qquad (a) \quad \frac{d^{2N}}{d\rho^{2N}} q(\rho)$$

exist for some N, and

$$(b) \quad \int_{\lambda=0}^{\infty} |Q(\lambda)| \lambda^{2N+2} \quad \text{exists.}$$

These two conditions constitute the smoothness requirements for this development.

Under these assumptions the λ integration in Eq. (B-5) can be performed to yield

$$(C-7) \qquad \Delta U(\rho, \xi, \tau) = \int_{\tau'=0}^{\tau} \theta_3 \left(\frac{\xi}{2}, \, i\pi\tau' \right) I(\rho, \tau')$$

where

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$$I(\rho,\tau) = q(\rho) + \sum_{n=1}^{N} \frac{\tau^n}{n!} \left(\frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} \right)^n q(\rho) + R_a$$

with
$$R_a \leqq \frac{\tau^{N+1}}{(N+1)!} \int_{\lambda=0}^{\infty} |Q(\lambda)| \, \lambda^{2N+3} \quad .$$

It is noted that R_a is $o(\tau^N)$ as $\tau \to 0$. The final result, an asymptotic expansion to N terms as $\tau \to 0$ of $\Delta U(\rho,\xi,\tau)$, is given by the expression

$$(C\text{-}8) \qquad \Delta U(\rho,\xi,\tau) = q(\rho)\, \psi_0(\xi,\tau) + \sum_{n=1}^{N} \frac{\psi_n(\xi,\tau)}{n!} \left(\frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} \right)^n q(\rho)$$

$$+ R_b$$

where
$$R_b \leq \frac{\psi_{N+1}(\xi,\tau)}{(N+1)!} \int_{\lambda=0}^{\infty} |Q(\lambda)| \, \lambda^{2N+3} , \quad 0 \leq \rho < \infty \quad ,$$

and where $\psi_n(\xi,\tau)$ is defined by

$$(C\text{-}9) \qquad \psi_n(\xi,\tau) = \int_{\tau'=0}^{\tau} (\tau')^n \, \theta_3 \left(\frac{\xi}{2} , i\pi\tau' \right) \qquad .$$

First, it is noted that the function, $\psi_0(\xi,\tau)$, is the temperature response for the flat plate with a uniform input over the entire front surface, except for a multiplicative constant:

$$(C\text{-}10) \qquad \psi_0(\xi,\tau) = \tau + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos n\pi\xi}{n^2} \left(1 - e^{-n^2 \pi^2 \tau} \right)$$

$$= \tau + \frac{1}{2} \xi^2 - \xi + \frac{1}{3} - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos n\pi\xi}{n^2} e^{-n^2 \pi^2 \tau} \quad .$$

The summation, $\displaystyle\sum_{1}^{\infty} \frac{\cos n\pi\xi}{n^2}$, is given by Jolley.[13] It should be noted that the integrals in Eq. (C-9) can be expressed in terms of incomplete gamma functions.

Thus the leading term in the small-time expansion is the product of the local heat flux input multiplied by the response for the flat plate with a uniform input over the entire surface. This particular approximation is valid on the surface as well as interior to the slab. Thus it is concluded that for time sufficiently small and the input flux density sufficiently smooth (as prescribed by conditions (a) and (b)), the radial diffusion is negligible. A similar result will be presented in the next section with the smoothness requirement weakened. However, the flat plate - uniform input response approximation will hold only on the input surface and at the center of the front surface.

D. The Green's Function

In this section the Green's function will be examined. Several expansions will be presented, some of which converge well for τ small and some of which converge well for τ large.

Some additional general asymptotic results for the temperature response as $\tau \to 0$ are given. The requirements on the input flux density, $q(\rho)$, are not as strong as they were in Section C.

To begin, the integration of Eq. (B-8) with respect to λ can be performed.[44] The result is given by

$$(D-1) \qquad G(\rho, \rho', \xi, \tau) = \int_{\tau'=0}^{\tau} \frac{1}{2\tau'} \theta_3\left(\frac{\xi}{2}, i\pi\tau'\right) I_0\left(\frac{\rho\rho'}{2\tau'}\right)$$

$$\exp\left(-\left\{\frac{\rho^2 + \rho'^2}{4\tau'}\right\}\right) \quad ,$$

where $I_0(x)$ is the modified Bessel function of the first kind. This expression may now be put in a more compact and in some respects a more useful form by introducing the integral form for the Bessel function:[45]

$$(D-2) \qquad I_0\left(\frac{\rho\rho'}{2\tau'}\right) = \frac{1}{\pi} \int_{\theta=0}^{\pi} e^{-\frac{\rho\rho'}{2\tau'}\cos\theta} d\theta$$

Using this representation one arrives at the following expression:

$$(D-3) \qquad G(\rho, \rho', \xi, \tau) = \frac{1}{2\pi} \int_{\theta=0}^{\pi} \int_{\tau'=0}^{\tau} \frac{1}{\tau'} \theta_3\left(\frac{\xi}{2}, i\pi\tau'\right) \exp\left(\frac{-|\overline{\rho} - \overline{\rho}'|^2}{4\tau'}\right)$$

where

$$|\overline{\rho} - \overline{\rho}'|^2 = \rho^2 + \rho'^2 - 2\rho\rho' \cos\theta \qquad .$$

Making use of Watson's suggestion for studying integrals containing a product of two Bessel functions of the same order, [16] as in Eq. (B-8), it is seen that [17]

$$(D-4) \qquad J_0(\lambda\rho) J_0(\lambda\rho') = \frac{1}{\pi} \int_{\theta=0}^{\pi} J_0(\lambda|\overline{\rho} - \overline{\rho}'|)$$

where the quantity, $|\overline{\rho} - \overline{\rho}'|$, is defined in Eq. (D-3), above.

Using this representation in Eq. (B-8) the expression for the Green's function becomes

$$(D-5) \qquad G(\rho, \rho', \xi, \tau) =$$

$$\frac{1}{\pi} \int_{\theta=0}^{\pi} \int_{\tau'=0}^{\tau} \int_{\lambda=0}^{\infty} \theta_3\left(\frac{\xi}{2}, i\pi\tau'\right) J_0(\lambda|\overline{\rho} - \overline{\rho}'|) e^{-\lambda^2\tau'} \lambda \qquad .$$

If one now performs the λ-integration the result is identical to Eq. (D-3). [18] The next logical attempt to simplify any of the above three representations would be to perform the τ'-integration. However, before this a short comment on the two expansions of the theta function (Eq. (B-6a) or (B-6b)) is necessary.

If the "long-time" expansion, Eq. (B-6a), is used in either Eq. (D-1) or Eq. (D-3), or if the short-time expansion, Eq. (B-6b), is used in Eq. (D-5), then the resulting integrals are of the form

$$(D-6) \qquad \int_{\tau'=0}^{\tau} q(\tau') \; e^{-\left(a\tau' + \frac{b}{\tau'}\right)}$$

It has not been possible to express these integrals in any simple fashion. The factor

$$e^{-\left(a\tau' + \frac{b}{\tau'}\right)} \qquad ,$$

has the form of a generating function for Bessel functions, but as yet it has not been possible to make use of this fact.

Therefore, the expansion of the theta function which is to be used in each of the above equations is determined by this restriction.

If the short-time expansion of the theta function is substituted in Eq. (D-1) the term-by-term integration can be performed only if the Bessel function is also expanded in its series. The resulting expression is very clumsy, and because of this complexity it has not been possible to determine the range of convergence of the series. It simplifies for special cases, but these special cases can be obtained by more direct methods.

The result is given by the expression:

$$(D-7) \qquad G(\rho, \rho', \xi, \tau) = \frac{1}{2\sqrt{\pi}} \sum_{\substack{m=0 \\ n=0 \\ \xi = \pm 1}}^{\infty} \frac{\epsilon_n}{(m!)^2} \left(\frac{\rho\,\rho'}{\rho^2 + \rho'^2 + 4(n + \xi\,\xi/2)^2}\right)^{2m} \cdot$$

$$\cdot \frac{1}{\sqrt{\rho^2 + \rho'^2 + 4(n + \xi\,\xi/2)^2}} \cdot \Gamma\left(2m + \tfrac{1}{2} , \frac{\rho^2 + \rho'^2 + 4(n + \delta\,\xi/2)^2}{4\tau}\right) ,$$

where $\qquad \Gamma(a, x) = \displaystyle\int_{x}^{\infty} t^{a-1} e^{-t}$ is the incomplete gamma function. [19]

For the special case $\rho = \xi = 0$, which is of interest in the case of focused inputs as mentioned in the introduction, the above expression simplifies to the following:

(D-8)
$$G(0, \rho', 0, \tau) = \frac{1}{\sqrt{\pi}} \frac{1}{\rho'} \Gamma\left(\frac{1}{2}, \frac{\rho'^2}{4\tau}\right)$$

$$+ \frac{2}{\sqrt{\pi}} \sum_{n=1}^{N} \frac{1}{\sqrt{\rho^2 + 4n^2}} \Gamma\left(\frac{1}{2}, \frac{\rho'^2 + 4n^2}{4\tau}\right)$$

$$+ R_3 \quad \text{where}$$

$$R_3 < \frac{\pi \sqrt{\pi \tau}}{6} e^{-1/\tau} \quad , \qquad N = 0$$

$$< \sqrt{\frac{\tau}{\pi}} e^{-(N+1)^2/\tau} \cdot \frac{1}{N} , \quad N \geq 1 \quad .$$

The result in Eq. (D-8) also follows from Eq. (D-3), by substituting the theta function from Eq. (B-6b). The gamma function, $\Gamma(\frac{1}{2}, x)$, is related to the incomplete error function. The relationship is given by

(D-9)
$$\Gamma\left(\frac{1}{2}, x\right) = \sqrt{\pi} \; \text{erfc}\,(\sqrt{x})$$

where
$$\text{erfc}\,(\sqrt{x}) = \frac{2}{\sqrt{\pi}} \int_{t=\sqrt{x}}^{\infty} e^{-t^2} \qquad .$$

It will be recalled that the incomplete error function enters into the representation for the small-time solution of the flat-plate response for the case of uniform input over the entire surface.[7]

The flat-plate response with a uniform input over the entire surface can be deduced from the above expression by letting $a \rightarrow \infty$ while $Q/\pi a^2$ is held constant.

An asymptotic expansion to one term in terms of $\sqrt{\tau}$ as $\tau \to 0$ can be obtained from Eq. (D-8). The result again displays the local nature of the response for time sufficiently small. From Eq. (D-8) one has

$$\text{(D-10)} \qquad \Delta U(0,0,\tau) = \frac{1}{\sqrt{\pi}} \int_{\rho'=0}^{\infty} q(\rho')\, \Gamma\left(\frac{1}{2}, \frac{\rho'^2}{4\tau}\right) + R_2 ,$$

where

$$R_2 < \frac{Q}{\ell k} \cdot \frac{\sqrt{\pi\,\tau}}{12}\, e^{-1/\tau} .$$

The integral in Eq. (D-10) is $O(\sqrt{\tau})$ as $\tau \to 0$; thus for sufficiently small time it will afford a good approximation. In general, however, the integration in Eq. (D-10) cannot be carried out in closed form.

It is possible, however, to obtain an asymptotic expansion of $\Delta U(0,0,\tau)$ as $\tau \to 0$ to one term under the following assumptions:

(a) $q(\rho) = q(o) + O(\rho)$ as $\rho \to 0$
(b) $q(\rho) < M$, a positive constant, for all ρ .

Under these assumptions, which are the smoothness requirement for this case, one arrives at the result

$$\text{(D-11)} \qquad \Delta U(0,0,\tau) = 2\sqrt{\frac{\tau}{\pi}}\, q(o) + O(\tau) \qquad \text{as } \tau \to 0 .$$

This result states that for τ sufficiently small the response at $\rho = \xi = 0$ is given by the local input multiplied by the small-time response of the plate to a uniform input over the entire front surface. This is the result one would expect on a physical basis. This result is almost the same as that deduced in the previous section in Eq. (C-5). And for $\rho = 0$ and τ sufficiently small, the one-term expansion of Eq. (C-8) is the same as (D-11).

Next, attention is focused on the integration of Eq. (D-3). For this expression the small-time expansion of the theta function is again used. The term-by-term integration yields the following expression:

$$(D\text{-}12) \qquad G(\rho, \rho', \xi, \tau) = \frac{1}{\pi \sqrt{\pi}} \int_{\theta=0}^{\pi} \left[\frac{\Gamma\left(\frac{1}{2}, \frac{\left|\overline{\rho} - \overline{\rho}'\right|^2 + \xi^2}{4\tau}\right)}{\sqrt{\left|\overline{\rho} - \overline{\rho}'\right|^2 + \xi^2}} \right.$$

$$+ \sum_{n=1}^{N} \frac{\Gamma\left(\frac{1}{2}, \frac{4(n-\xi/2)^2 + \left|\overline{\rho} - \overline{\rho}'\right|^2}{4\tau}\right)}{\sqrt{4(n-\xi/2)^2 + \left|\overline{\rho} - \overline{\rho}'\right|^2}}$$

$$\left. + \sum_{n=1}^{N} \frac{\Gamma\left(\frac{1}{2}, \frac{4(n+\xi/2)^2 + \left|\overline{\rho} - \overline{\rho}'\right|^2}{4\tau}\right)}{\sqrt{4(n+\xi/2)^2 + \left|\overline{\rho} - \overline{\rho}'\right|^2}} \right]$$

$$+ R_3,$$

where the remainder R_3 is bounded for all ρ by

$$R_3 \leq \frac{1}{\sqrt{\pi}} \cdot \frac{1}{N - \xi/2} \cdot \exp\left(-\left\{\frac{(N+1-\xi/2)^2}{\tau}\right\}\right)$$

$$+ \frac{1}{\sqrt{\pi}} \cdot \frac{1}{N + \xi/2} \cdot \exp\left(-\left\{\frac{(N+1+\xi/2)^2}{\tau}\right\}\right) \qquad \begin{array}{c} 0 \leq \xi \leq 1 \\ N \geq 1 \end{array}$$

$$R_3 < \frac{\pi}{6} \sqrt{\pi\tau} \; e^{-1/\tau}, \qquad N = 0, \qquad \xi = 0$$

$$R_3 < \frac{\pi}{4} \sqrt{\pi\tau} \left\{\frac{1}{3} e^{-(1-\xi/2)^2/\tau} + e^{-(1+\xi/2)^2/\tau}\right\}$$

$$\begin{array}{c} N = 0 \\ 0 < \xi \leq 1. \end{array}$$

This result is quite similar to that in Eq. (D-7), except that the summation on "m" in Eq. (D-7) has been replaced by an integration on θ in Eq. (D-12), and it has been possible to deduce a remainder for this last expansion. This approach then provides a method for a partial summing of Eq. (D-7). Furthermore, using Eq. (D-12), with time sufficiently small the results in Eq. (D-10) can be generalized to yield

$$(D-13) \qquad \Delta U(\rho, \xi, \tau) = \int_{\rho'=0}^{\infty} q(\rho') \, \rho' \left[\frac{1}{\pi \sqrt{\pi}} \int_{\theta=0}^{\pi} \frac{\Gamma\left(\frac{1}{2}, \frac{|\overline{\rho} - \overline{\rho}'|^2 + \xi^2}{4\tau}\right)}{\sqrt{|\overline{\rho} - \overline{\rho}'|^2 + \xi^2}} \right] + R_3$$

where the remainder is bounded for all ρ by

$$R_3 < \frac{Q}{\ell k} \frac{\sqrt{\pi \tau}}{12} e^{-1/\tau} \qquad , \qquad \xi = 0$$

$$R_3 < \frac{Q}{\ell k} \frac{\sqrt{\pi \tau}}{8} \left\{ \frac{1}{3} e^{-(1-\xi/2)^2/\tau} + e^{-(1+\xi/2)^2/\tau} \right\} .$$

The result in Eq. (D-13) reduces to Eq. (D-10) for $\rho = 0$ since the integrand becomes independent of θ and the θ-integration can be performed. In this case the dependence on ξ can be retained, and in this sense it is more general than Eq. (D-10). An asymptotic expansion can be deduced similar to that given in Eq. (D-11). Equation (D-13) is written for $\rho = 0$ as follows:

$$(D-14) \qquad \Delta U(0, \xi, \tau) = \frac{1}{\sqrt{\pi}} \int_{\rho'=0}^{\infty} q(\rho') \frac{\rho'}{\sqrt{\xi^2 + \rho'^2}} \Gamma\left(\frac{1}{2}, \frac{\rho'^2 + \xi^2}{4\tau}\right) + R_3$$

where R_3 is bounded as above in Eq. (D-13).

The asymptotic development of Eq. (D-14) yields (under the same assumptions as those for Eq. (D-11)) the result:

$$(D-15) \qquad \Delta U(0, \xi, \tau) \sim q(o) \; 2 \sqrt{\frac{\tau}{\pi}} \left[e^{-\xi^2/4\tau} - \frac{\xi}{2\sqrt{\tau}} \; \Gamma \left(\frac{1}{2}, \frac{\xi^2}{4\tau} \right) \right] + O(\tau)$$

as $\tau \to 0$.

At the front surface, $\xi = 0$, this reduces to the result given in Eq. (D-11). Again the local nature of the response for small time is emphasized. Furthermore, it is seen that this approximation is independent of the thickness of the slab, ℓ, while the remainder is not.

The next approach is to perform the τ' integration in Eq. (D-5). This yields the following result:

$$(D-16) \qquad G(\rho, \rho', \xi, \tau) = \frac{1}{\pi} \int_{\theta=0}^{\pi} \int_{\lambda=0}^{\infty} \left[\frac{1 - e^{-\lambda^2 \tau}}{\lambda^2} \right.$$

$$+ 2 \sum_{n=1}^{\infty} \frac{\cos n\pi\xi}{\lambda^2 + n^2 \pi^2} \left\{ 1 - e^{-(\lambda^2 + n^2 \pi^2)\tau} \right\} \bigg] J_0(\lambda |\overline{\rho} - \overline{\rho}'|) \lambda.$$

It has unfortunately not been possible to perform the following integration:

$$(D-17) \qquad R_4 = 2 \int_{\lambda=0}^{\infty} \sum_{n=1}^{\infty} \frac{\cos n\pi\xi}{\lambda^2 + n^2 \pi^2} e^{-(\lambda^2 + n^2 \pi^2)\tau} J_0(\lambda |\overline{\rho} - \overline{\rho}'|) \lambda.$$

However, an upper bound can be obtained on this remainder and the result is an expansion which is accurate for τ sufficiently large. The bound, valid for all $0 \le \xi \le 1$ and $0 \le \rho < \infty$, $0 \le \rho' < \infty$, is given by

$$(D-18) \qquad R_4 < \frac{1}{6} \cdot \frac{e^{-\pi^2 \tau}}{\tau} \qquad .$$

When the exponential factor is not present it is possible to perform the λ-integration[20] in the second term of Eq. (D-16).

This yields the following expansion:

$$
\text{(D-19)} \qquad G(\rho, \rho', \xi, \tau) = \frac{1}{\pi} \int_{\theta=0}^{\pi} \int_{\lambda=0}^{\infty} \left[\frac{1 - e^{-\lambda^2 \tau}}{\lambda^2} \right] J_0(\lambda \, |\overline{\rho} - \overline{\rho'}|) \lambda
$$

$$
+ \frac{1}{\pi} \int_{\theta=0}^{\pi} 2 \sum_{n=1}^{\infty} \cos n\pi\xi \, K_0 (n\pi \, |\overline{\rho} - \overline{\rho'}|) + R_4 \, .
$$

It is also possible to rewrite this last sum, making use of the addition theorem,[21] as

$$
\text{(D-20)} \qquad \frac{1}{\pi} \int_{\theta=0}^{\pi} K_0(n\pi \, |\overline{\rho} - \overline{\rho'}|) =
$$

$$
\begin{array}{ll}
I_0(n\pi\rho) \, K_0(n\pi\rho') & \rho < \rho' \\[2mm]
I_0(n\pi\rho') \, K_0(n\pi\rho) & \rho > \rho' \quad .
\end{array}
$$

It appears that the only way to handle the first integral is to make use of the equality

$$
\text{(D-21)} \qquad \frac{1 - e^{-\lambda^2 \tau}}{\lambda^2} = \int_{\tau'=0}^{\tau} e^{-\lambda^2 \tau'} \qquad \qquad .
$$

By making use of this, both the λ-integration and the τ'-integration can be carried out in terms of the incomplete gamma function, $\Gamma(0, x)$, or the exponential integral.

Incorporating these suggestions and using Eq. (D-20) in Eq. (D-19), one arrives at the following expression for the Green's function, which is useful for τ large:

(D-22) $G(\rho, \rho', \xi, \tau) = \dfrac{1}{2\pi} \displaystyle\int_{\theta=0}^{\pi} \Gamma\left(0, \dfrac{|\bar{\rho}-\bar{\rho}'|^2}{4\pi}\right) +$

$$+ \begin{cases} \displaystyle\sum_{n=1}^{\infty} \cos n\pi\,\xi I_0(n\pi\rho)\, K_0(n\pi\rho'), \quad \rho < \rho' \\[2em] \displaystyle\sum_{n=1}^{\infty} \cos n\pi\,\xi I_0(n\pi\rho')\, K_0(n\pi\rho), \quad \rho > \rho' \end{cases}$$

$$+ R_4, \text{ where } R_4 < \dfrac{1}{6}\,\dfrac{e^{-\pi^2\tau}}{\tau} \qquad \begin{array}{l} 0 \le \rho < \infty \\[0.3em] 0 \le \rho' < \infty \\[0.3em] 0 \le \xi \le 1 \,. \end{array}$$

It has not been possible to deduce any general asymptotic results for τ large similar to those in Eqs. (D-11) and D-15) for τ small.

In this section several expansions have been presented for the Green's function. Some of these converge quickly for τ small, and some of them for τ large. In addition, some general asymptotic results for the temperature response as $\tau \to 0$ have been given. The requirements on $q(\rho)$ needed to obtain these results are not as strong as in the case of the development in Section C.

E. Gaussian Input

The thermal response to the Gaussian input is discussed in this section. Two small-time expansions for the temperature at the center of the spot, $\rho = \xi = 0$, are given and their relative accuracy is compared. In addition the criteria necessary to use the asymptotic expansion presented in Eq. (C-8) are satisfied for this input flux density. For the parameter $4\tau/\eta_0^2 < 1$, this yields a convergent series which can be summed to give a form convenient for numerical computation. This result can also be obtained by other techniques. The asymptotic expansion deduced from Eq. (C-8) is of course not restricted to the spot $\rho = \xi = 0$.

The Gaussian input is defined as follows:

(E-1) $\qquad q(\rho) = \dfrac{Q}{\pi \ell k} \cdot \dfrac{1}{r_i^2} \cdot \exp(-\rho^2 / \eta_o^2) \, , \ \rho \geq 0$

where Q is the total heat flux input and $\eta_o = a/\ell$ where "a" is the "radius" of the spot. The transformed input, $Q(\lambda)$, is given by a function of the same form,[2][2]

(E-2) $\qquad Q(\lambda) = \dfrac{Q}{2\pi \ell k} \cdot \exp(-\lambda^2 \eta_o^2 / 4).$

The fact that the parameter λ enters the expression for the temperature field in the same manner as the factor $\exp(-\lambda^2 t')$ in Eq. (B-5) makes this input particularly amenable to analysis. Here is the first divergence from the rather general results obtained in the last two sections. It is a fruitful divergence because of the special form of the heat flux input. The short-time expansion of the theta function given in Eq. (B-6b) is going to be used with the $F(\rho, \xi, \tau; \lambda)$ form. This will, of course, limit the results to the case $\xi = 0$ and a one-term expansion. This is because of the difficulties presented by integrals of the form

$$\int_{\tau'= 0}^{\tau} g(\tau') \, e^{\left(a\tau' + \frac{b}{\tau}\right)} :$$

as was discussed above in Section D. An upper bound on the remaining terms in determined. The result is given by the expression

(E-3) $\qquad \Delta U(0,0,\tau) = \dfrac{Q}{ak} \left[\dfrac{1}{\pi\sqrt{\pi}} \tan^{-1} \sqrt{\dfrac{4kt}{a^2}} \ \right] + R_t$

where

$$R_5 < \frac{Q}{\ell k} \frac{\sqrt{\pi \tau}}{12} \left(\frac{4kt/a^2}{1 + 4kt/a^2} \right) e^{-1/\tau} \quad .$$

There are several things to note about this expression. First, for τ small, this approximation is best, that is the error bound is controlled by the term $e^{-1/\tau}$. Second, the approximation itself does not depend on the thickness of the slab, but error, R_5, does. Third, for t fixed the approximation becomes better as the thickness of the slab increases. This statement is somewhat redundant since this conclusion follows from the first. However, the conclusion in this form emphasizes the "small-time" aspect of the approximation. Furthermore, as $\ell \rightarrow \infty$ this response approaches that of the semi-infinite body at $\rho = \xi = 0$ since the remainder $R_5 \rightarrow 0$ as $\ell \rightarrow \infty$. A plot of this approximation is shown in Fig. 2; while the plot appears to say that the temperature is bounded as $t \rightarrow \infty$, this is not the case since, as the remainder indicates the approximation eventually becomes very poor as t increases with the other parameters fixed.
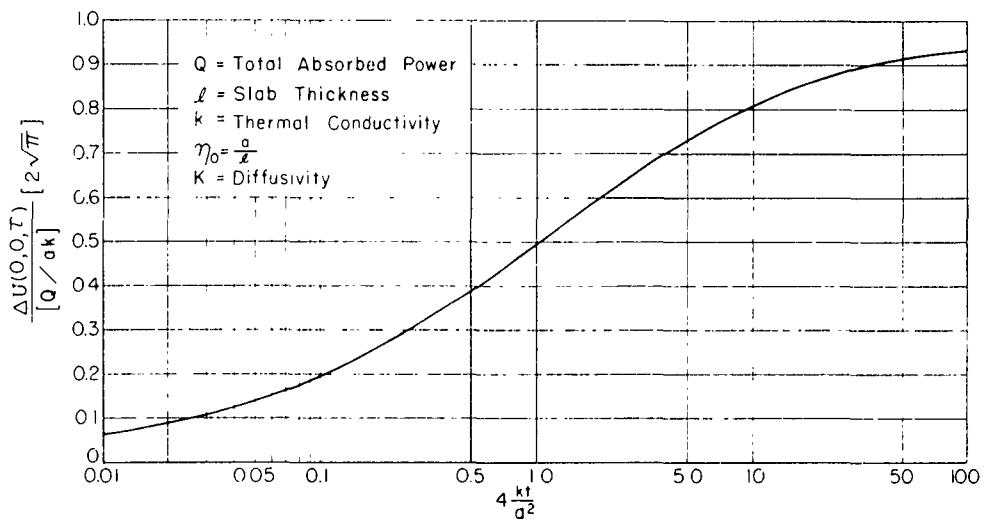
Fig. 2. The response to the Gaussian input as given by the arc-tan representation.

The second expansion, deduced from the large-time theta function expansion given in Eq. (B-6a) and the $F(\rho, \xi, \pi; \lambda)$ form, is convergent for all τ. In contrast to the arc-tan representation, the first term of this expansion does depend on the thickness of the slab, ℓ. The expression for the remainder shows that the remainder is again small for $4kt/a^2$ small, but by contrast to the arc-tan representation, the representation is best when the parameter η_0 is large. The representation is given by

(E-4)
$$\Delta U(0, \xi, \tau) = \frac{Q}{2\pi \ell k} \left[\ell n \sqrt{\frac{1 + 4Kt}{a^2}} \right.$$

$$+ \sum_{n=1}^{N} \cos n\pi \xi \left(\Gamma\left(0, \frac{n^2 \pi^2 \eta_0^2}{4}\right) - \Gamma\left(0, \frac{n^2 \pi^2 \eta_0^2}{4} \left\{ 1 + \frac{4Kt}{a^2} \right\} \right) \right)$$

$$\left. \cdot \exp \frac{n^2 \pi^2 \eta_0^2}{4} \right] + R_6$$

where

$$R_6 \leq \frac{Q}{\ell k} \cdot \frac{2}{\pi^3 \eta_0^2} \cdot \left[\frac{1 - \exp(-N^2 \pi^2 \tau)}{N} + \right.$$

$$\left. 2\pi \sqrt{\pi \tau} \text{ erfc } N\pi \sqrt{\tau} \right], \quad N \geq 1$$

$$\leq \frac{Q}{\ell k} \cdot \frac{2}{\pi \eta_0^2} \cdot \sqrt{\frac{\tau}{\pi}}, \quad N = 0.$$

From the above expression for the remainder it can be shown that for all τ the remainder can be made arbitrarily small by increasing the number of terms, N, in the expansion. However, as N becomes large the remainder vanishes only as 1/N and the convergence is very slow. However,

in contrast to the arc-tan representation, arbitrairly good accuracy can in principle be obtained from this expansion for all τ. Finally, the accuracy of this representation when only one term is retained is not as good as that obtainable from the arc-tan representation, as an examination of the remainders show. A graph of the logrithm representation is given in Fig. 3. for two values of η_o, using the log term plus the first term in the series.
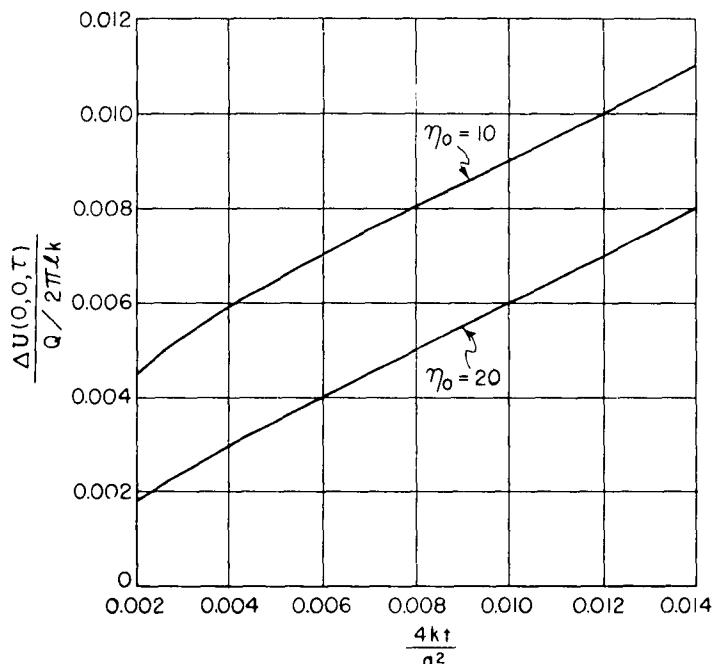

Fig. 3. This is the response to the Gaussian input as given by the log representation using the log term plus the first term in the series. The response is plotted for two values of η_o.

This graph shows that for τ and the total heat input fixed, the temperature decreases as η_o increases. This is the expected result.

The requirements necessary to use the asymptotic expansion in Eq. (C-8) are fulfilled by the Gaussian input. The first three terms in

the expansion are given by

$$\Delta U(\rho, \xi, \tau) = q(\rho) \left[\psi_o(\xi; \tau) + \right.$$

$$\frac{4}{\eta_o^2} \left[\left(\frac{r}{a} \right)^2 - 1 \right] \psi_1(\xi, \tau)$$

$$+ \left(\frac{4}{\eta_o^2} \right)^2 \left[\left(\frac{r}{a} \right)^4 - 4 \left(\frac{r}{a} \right)^2 + 2 \right] \psi_2(\xi, \tau) \left. \right] + R_b,$$

where the remainder is bounded by

$$R_b \leq \frac{Q}{\ell k} \cdot \frac{1}{\pi \eta_o^2} \cdot \left(\frac{4Kt}{a^2} \right)^4 \cdot \psi_o(\xi, \tau),$$

for all ρ. This may also be written for the case of the Gaussian input as follows.[12]

(E-5) $\Delta U(\rho, \xi, \tau) = q(\rho) \sum_{n=0}^{N} (-1)^n \left(\frac{4}{\eta_o^2} \right)^n \psi_n(\xi, \tau) L_n \left(\frac{\rho^2}{\eta_o^2} \right) + R_6$

where

$$\psi_n(\xi, \tau) = \int_{\tau'=0}^{\tau} (\tau')^n \theta_3 \left(\frac{\xi}{2}, i \pi \tau' \right),$$

$L_n(x)$ = Laguerre polynomials,[23] and

$$R_b \le q(\rho) \left(\frac{4}{\eta_o^2}\right)^{N+1} L_{N+1}\left(\frac{\rho^2}{\eta_o^2}\right) \psi_{N+1}(\xi, \tau), \ N \ge 0.$$

The expression for the upper bound can be simplified by use of the fact that

$$(E-6) \qquad \psi_{N+1}(\xi, \tau) \le \tau^{N+1} \psi_o(\xi, \tau)$$

and

$$L_n\left(\frac{\rho^2}{\eta_o^2}\right) \le e^{\rho^2/\eta_o^2},$$

where $\psi_o(\xi, \tau)$ is given by Eq. (C-10).

One last representation for the temperature fields will be given. It can be obtained by a variety of techniques. One technique is by introducing the integral definition of $\psi_n(\xi, \tau)$ into Eq. (E-5) and then noting that[24]

$$(E-7) \qquad \sum_{n=0}^{\infty} z^n L_n(x) = \frac{1}{1-z} \exp \frac{xz}{z-1}, \ |z| < 1.$$

Another method for arriving at the same result is by direct integration of Eq. (B-5):

$$(E-8) \qquad \Delta U(\rho, \xi, \tau) = \frac{Q}{2\pi \ell k} \int_{\tau'=0}^{\tau} \theta_3\left(\frac{\xi}{2}, i\pi\tau'\right) \int_{\lambda=0}^{\infty} e^{-(\tau'+\frac{\eta_o^2}{4})\lambda^2} J_o(\lambda\rho)\lambda.$$

The integration with respect to λ can be performed[17] to yield the following result:

$$(E-9) \qquad \Delta U(\rho, \xi, \tau) = \frac{Q}{\pi \ell k} \int_{\tau'=0}^{\tau} \theta_3\left(\frac{\xi}{2}, i\pi\tau'\right) \left(\frac{1}{4\tau' + \eta_o^2}\right) \exp\left(\frac{-\rho^2}{4\tau' + \eta_o^2}\right).$$

This form of the response seems most useful if a numerical computation is required. The term $q(\rho)$ can be factored from this expression to yield the result indicated by the sum in Eq. (E-7).

F. The Uniform Input

In this section the temperature fields due to a uniform input on a circular spot of finite radius are presented. The input is defined in Eq. (F-1) below. Because of the discontinuity in $q(\rho)$, the results for this case are not as abundant as they were for the case of the Gaussian input.

One of the results is an extension of Oosterkamp's work.[3] It is extended to include results for $0 \leq \xi \leq \ell$ and $\rho = 0$, while his results held only for $\rho = \xi = 0$. Included is an upper bound on the remainder which serves to give a precise meaning to the approximation. Graphical information is also included.

The long-time Green's function, defined in Eq. (D-22), is used to obtain the response for the uniform input. A simple explanation in terms of the effect of radial diffusion is given. Graphical information is also included.

The results of this section also show that when the spot size increases to infinity and the input flux density is held constant, the temperature response approaches that of the flat plate with a uniform input over the entire front surface. This is of course to be expected.

The uniform input is defined by the expression

$$(F-1) \qquad q(\rho) = \frac{Q}{\pi a^2} \cdot \frac{1}{k} , \quad \rho < \eta_0$$

$$= 0 , \qquad \rho > \eta_0$$

where a   is the radius of the spot and

$$\eta_o = \frac{a}{\ell} \quad .$$

The transformed input is given by the expression[25]

$$(F-2) \qquad Q(\lambda) = \frac{Q}{\pi a^2} \cdot \frac{1}{k} \cdot \eta_o \frac{J_1(\lambda \eta_o)}{\lambda} \quad .$$

The Green's function, $G(\rho, \rho', \xi, \tau)$ seems to be the most useful for this input because the integrand of the expression,

$$(F-3) \qquad \Delta U(\rho, \xi, \tau) = \int_{\lambda=0}^{\infty} Q(\lambda) \, F(\rho, \xi, \tau; \lambda) \lambda \, ,$$

contains the product of two Bessel functions of different order and argument, and these, as mentioned before, are the most difficult to handle. The only possible approach using the representation in Eq. (F-3) is to make use of the expansion[26]

$$(F-4) \qquad J_1(\lambda \eta_o) J_0(\lambda \rho) =$$

$$\frac{\lambda \eta_o}{2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{\lambda \rho}{2} \right)^{2m} {}_2F_1 \left(-m, -m; 2; \eta_o^2/\rho^2 \right)$$

$$\rho > \eta_o$$

$$\frac{\lambda \eta_o}{2} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!} \left(\frac{\lambda \eta_o}{2} \right)^{2m} {}_2F_1 \left(-m, -1-m; 1; \rho^2/\eta_o^2 \right)$$

$$\rho < \eta_o .$$

The functions $_2F_1(-m, a;b;x^2)$ are hypergesnetric functions, [27] and because of the factor $(-m)$ are polynomials of degree m in x^2. This expansion allows one to perform the λ integration, but the integration on τ' cannot be carried out for $\xi = 0$. The special case $\rho = 0$ can be handled with the form given in Eq. (F-3) but will be obtained later with the use of the Green's function. This result is given by the expression

$$\text{(F-5)} \qquad \Delta U(0, \xi, \tau) = \frac{Q}{\pi a^2} \cdot \frac{\ell}{k} \cdot \int_{\tau'=0}^{\tau} (1 - e^{-\eta_0^2/4\tau'}) \, \theta_3 \left(\frac{\xi}{2}, i\pi\tau' \right) .$$

Before passing to the Green's function approach, several comments about this result are in order. First, it is recalled that the solution to the flat-plate problem with a uniform input over the entire surface is given by

$$\text{(F-6)} \qquad \Delta U = \frac{Q}{\pi a^2} \frac{\ell}{k} \int_{\tau'=0}^{\tau} \theta_3 \left(\frac{\xi}{2}, i\pi\tau' \right) .$$

This is precisely the result that one would obtain from Eq. (F-5) by holding $Q/\pi a^2$ constant and letting $a \to \infty$. Also for τ small the effect of the exponential term in the integral of Eq. (F-5) is small, and thus it is seen that when τ is small, the response of the temperature at $\rho = 0$ is that of the flat plate with a uniform input over the entire surface. This again indicates the local nature of the response for τ small.

If the first term in the small-time expansion of the theta function for $\xi = 0$ is retained and the integration performed, the result is the same as Oosterkamp's. It is not necessary, however, to require $\xi = 0$. The integration can be performed even for $\xi \neq 0$. Also an upper bound on the remainder can be computed. For $\xi = 0$ the result is given by

$$\text{(F-7)} \qquad \Delta U(0, 0, \tau) = \frac{Q}{\pi ak} \cdot \frac{1}{\sqrt{\pi}} \left[\sqrt{\frac{4\tau}{\eta_0^2}} - \frac{1}{2} \, \Gamma \left(-\frac{1}{2}, \frac{\eta_0^2}{4\tau} \right) \right] + R_2$$

where

$$R_2 < \frac{Q}{\ell k} \cdot \frac{\sqrt{\pi \tau}}{12} \, e^{-1/\tau} \quad .$$

Attention is called to the similarity of this result and the small-time approximation given in Eq. (D-15).

The result in Eq. (F-17) may also be rewritten as follows:

$$(F-8) \qquad \Delta U(0, 0, \tau) = \frac{Q}{\pi a k} \cdot \sqrt{\frac{4Kt}{a^2}} \cdot \left[\frac{1 - e^{a^2/4Kt}}{\sqrt{\pi}} + \right.$$

$$\left. \frac{a}{2\,Kt} \, erfc\left(\frac{a}{2\sqrt{Kt}} \right) \right] + R_2 \quad ,$$

which is closer to Oosterkamp's notation. A plot of this result is shown in Fig. 4. These formulas emphasize the "small-time" aspect of the approximation. Note that the approximation does not depend on the thickness of the slab (it is therefore the response of the semi-infinite slab since $R_2 \to 0$ as $\ell \to \infty$), but the bound on the remainder does depend on the slab thickness. The requirement that Eq. (F-7) or (F-8) be a good approximation is that

$$\tau = \frac{Kt}{\ell^2} \quad \text{be small.}$$

Thus for any fixed t (in seconds), the thickness of the slab can be increased so that the approximation in Eq. (F-7) or (F-8) offers an approximation as good as needed. Some typical values of the normalized remainder $R/[Q/\ell k]$, are given in Table I below.

TABLE I
Some typical values of the remainder
associated with Eqs. (F-7) and (F-8).

τ	$R/[Q/\ell k]$
.01	$5 \cdot 10^{-46}$
0.1	$2 \cdot 10^{-6}$
1.0	$5 \cdot 10^{-2}$
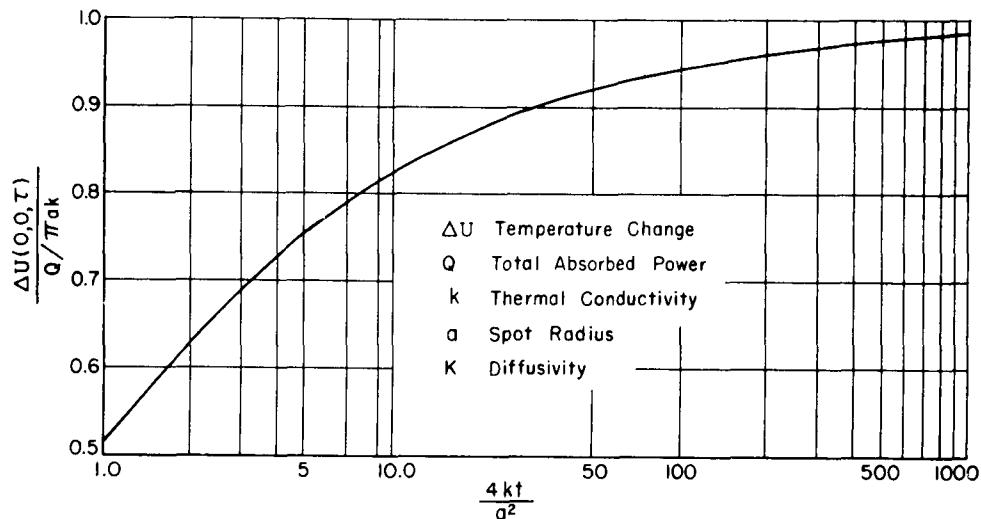
Fig. 4. This is the response to the uniform
stop as obtained by Oosterkamp.

For $\xi \neq 0$ the response is given by the following, which is quite similar
to the earlier result in Eq. (F-8):

$$(F-9) \quad \Delta U(0,\xi,\tau) = \frac{Q}{\pi ak}\left[\sqrt{\frac{4Kt}{a^2}} \left\{1 - e^{-\frac{a^2}{4Kt}}\right\} e^{-\frac{(z/a)^2}{4Kt/a^2}}\right.$$

$$+\sqrt{1+(z/a)^2} \ \ \mathrm{erfc}\left(\sqrt{\frac{1+(z/a)^2}{4Kt/a^2}}\right)$$

$$\left.-(z/a) \ \mathrm{erfc}\left(\frac{(z/a)}{\sqrt{4Kt/a^2}}\right)\right] + R_7$$

where

$$R_7 < \frac{Q}{\ell k} \cdot \frac{\sqrt{\pi\tau}}{4} \left[\frac{1}{6} \ \frac{e^{-(1-\xi/2)^2}}{\tau} + \frac{1}{2} \ \frac{e^{-(1+\xi/2)^2}}{\tau} \right] \quad ,$$

$$0 < \xi \leq 1.$$

Note again that there is no dependence on the thickness except in the remainder, and that as $\ell \to \infty$, $R_7 \to 0$. For $\xi = 0$ the above approximation reduces to the former case.

Now the long-time Green's function is used to obtain the temperature response for τ large. This Green's function is given in Eq. (D-22). In order to obtain this result, use is made of the two integrals:[28, 29]

$$\int_{\rho'=0}^{\rho} \rho' I_0(n\pi\rho') = \frac{\rho}{n\pi} \ I_1(n\pi\rho),$$

$$\int_{\rho'=\rho}^{\eta_0} \rho' K_0(n\pi\rho') = \frac{1}{n\pi} \left[\rho K_1(n\pi\rho) - \eta_0 K_1(n\pi\eta_0) \right].$$

The temperature fields for $\rho < \eta_0$ are given by

$$(F-10) \qquad \Delta U(\rho, \xi, \tau) = \frac{Q}{\pi a^2} \ \frac{\ell}{k} \ \frac{1}{2\pi} \int_{\rho'=0}^{\eta_0} \int_{\theta=0}^{\pi} \rho' \ \Gamma\left(0, \frac{|\vec{\rho}-\vec{\rho}'|^2}{4\tau}\right)$$

$$+ 2 \ \frac{Q}{\pi a^2} \ \frac{\ell}{k} \ \sum_{n=1}^{\infty} \frac{\cos n\pi\xi}{n\pi} \ [K_0(n\pi\rho)\rho I_1(n\pi\rho)$$

$$+ I_0(n\pi\rho)\{\rho K_1(n\pi\rho) - \eta_0 K_1(n\pi\eta_0)\}] + R_1 \ .$$

The temperature fields for $\rho > \eta_o$ are given by

(F-11)
$$\Delta U(\rho, \xi, \tau) = \frac{Q}{\pi a^2} \frac{\ell}{k} \frac{1}{2\pi} \int_{\rho'=0}^{\eta_o} \int_{\theta=0}^{\pi} \rho' \Gamma \left(0, \frac{|\bar{\rho} - \bar{\rho}'|^2}{4\tau} \right)$$

$$+ 2 \frac{Q}{\pi a^2} \frac{\ell}{k} \sum_{n=1}^{\infty} \frac{\cos n\pi \xi}{n\pi} {}'K_o(n\pi\rho) \eta_o I_1 (n\pi \eta_o) + R_1$$

where

$$|\bar{\rho} - \bar{\rho}'|^2 = \rho^2 + \rho'^2 - 2\rho\rho' \cos\theta$$

and

$$R_1 < \frac{Q}{\ell k} \frac{e^{-\pi^2 \tau}}{12\pi\tau} \quad .$$

Some typical values of the normalized remainder are given in Table II.

TABLE II
Some typical values of the normalized
remainder for Eqs. (F-10) and (F-11).

τ	$R_1/[Q/\ell k]$
0.1	.10
1.0	$.4 \cdot 10^{-3}$
10.	$.15 \cdot 10^{-5}$

Again attention is called to the fact that as a $\to \infty$ but with $Q/\pi a^2$ held constant, the above result (Eq. (F-10) must be used because of the restriction $\rho < \eta_o$) reduces to the response for the flat plate with a uniform input over the entire front surface, except for the term which contains both a ξ dependence and the time-dependence. This term has been

absorbed by the remainder. The last summation becomes independent of ρ since it is ρ multiplied by the Wronskian of $I_0(x)$, $K_0(x)$. The initial integration can be performed and is proportional to τ but is also independent of ρ.

Again the special case $\rho = 0$ can be given more attention. Setting $\rho = 0$ in Eq. (F-10) one arrives at the following result:

$$(F-12) \qquad \Delta U(0, \xi, \tau) = \frac{Q}{\pi a^2} \frac{\ell}{k} \tau \; \Phi \left(\frac{4Kt}{a^2} \right)$$

$$+ \frac{Q}{\pi a^2} \frac{\ell}{k} \left[\frac{1}{2} \xi^2 - \xi + \frac{1}{3} \right.$$

$$\left. - 2\eta_o^2 \sum_{n=1}^{\infty} \frac{\cos n\pi\xi}{n\pi\eta_o} K_1 (n\pi\eta_o) \right] + R_1$$

where

$$\Phi \left(\frac{4Kt}{a^2} \right) = \left(1 - e^{-\frac{a^2}{4Kt}} \right) + \frac{a^2}{4Kt} \; \Gamma \left(0, \; \frac{a^2}{4Kt} \right) \quad .$$

Again as $\eta_o \to \infty$ but $\frac{Q}{\pi a^2}$ remains fixed, this result reduces to the flat-plate response with a uniform heat flux density input over the entire surface.

Note that the first term in this response is "τ" multiplied by a factor in brackets. This factor

$$\Phi \left(\frac{4Kt}{a^2} \right) = \left[1 - e^{-\eta_o^2/4\tau} + \frac{\eta_o^2}{4\tau} \; \Gamma \left(0, \; \frac{\eta_o^2}{4\tau} \right) \right] ,$$

will be called the diffusion factor, and it carries the effect of not having a uniform input over the entire surface. Note that it is independent of the slab thickness. This factor is shown graphically in Fig. 5. This
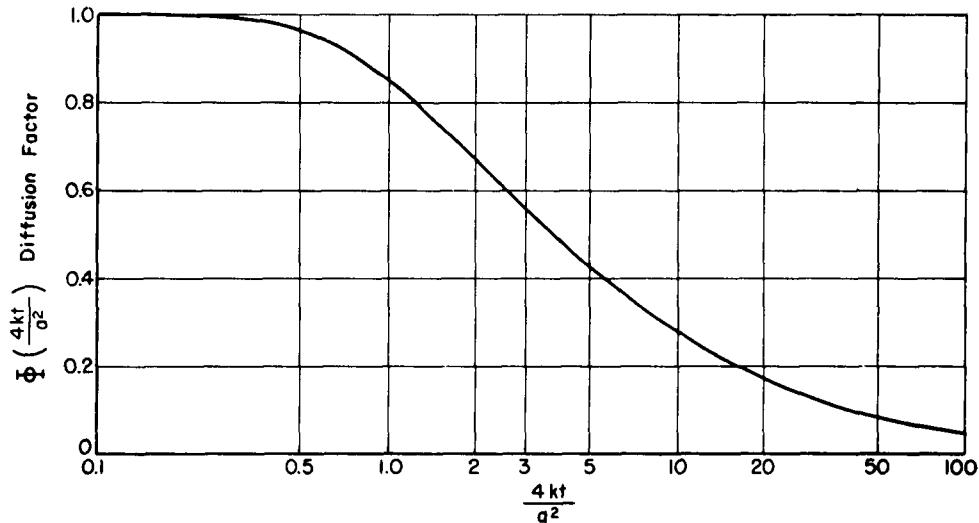
Fig. 5. This is a plot of the diffusion factor defined in Eq. (F-12) assoicated with the long-time response for the uniform spot.

figures shows that as "a" increases, for t fixed, the effect of radial diffusion at $\rho = 0$ decreases. Also it shows that as t or K increases for "a" fixed, the effect of radial diffusion at $\rho = 0$ also increases. The last term in the response of Eq. (F-12),

$$\left[\frac{1}{2} \, \xi^2 \, - \xi + \frac{1}{3} \, - 2\eta_o^2 \, \sum_{n=1}^{\infty} \frac{\cos n\pi \xi}{n\pi} \, K_1 \, (n\pi \eta_o) \, \right] ,$$

is independent of time, and as $\eta_o \rightarrow \infty$ the last sum vanishes and also the response. Thus the larger η_o, the less the effect of radial diffusion at $\rho = 0$. Fig. 6 displays this last sum for a range of η_o.

The graph contains an equation box:

$$0 < \frac{2}{\pi^2} \sum_{n=1}^{\infty} (\pi\eta_0) \frac{k_1(n\pi\eta_0)}{n} \leq \frac{1}{3}$$

Y-axis label: $\frac{2}{\pi^2} (\pi\eta_0) \sum \frac{k_1(n\pi\eta_0)}{n}$

X-axis label: $\pi\eta_0$
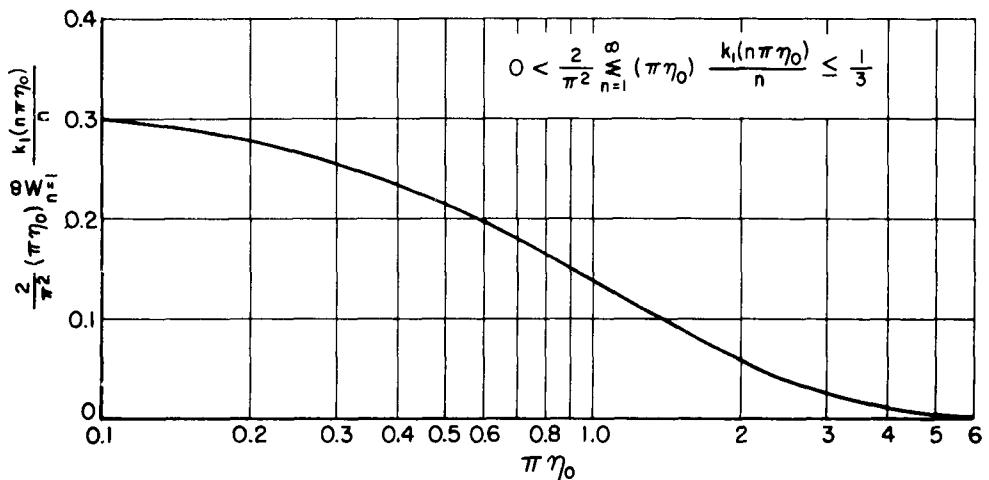
Fig. 6. The last summation in Eq. (F-12) is shown for $\xi = 0$ and $0.1 < \pi\eta_0 < 60$.

G. Conclusions

Transformation techniques are used to obtain several alternative representations of the solution in terms of the Green's function and its transform. Some representations converge well for time small, and some for time large.

Several asymptotic expansions are deduced for the small-time response of the temperature fields under various assumptions on the heat flux input density.

An extension of the results of Oosterkamp has been derived as well as a verification of the validity of the approximations through the derivation of an upper bound on the error.

Two specific inputs have been considered in detail: the Gaussian, and the uniform spot. These two inputs can serve as models of the heat flux input caused by such sources as the laser; the study of the thermal response is the initial step in a study of the thermal-elastic response of a slab to an input such as the laser.

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39 pp. including illustrations
OSURF Report 1107-14

The temperature fields are deduced for an
infinite metal slab of finite thickness produced
by a heat flux input which depends on the radial
coordinate. The formal solution is obtained
by means of transform techniques, and several
alternative approaches to their evaluation are
presented. Two heat flux inputs are considered
in detail. They are the Gaussian input defined
in Eq. (E-1), and the uniform input over a

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circular area of the front surface. Such
inputs might be used, for example, as
models for a study of the thermal-elastic
effects of a laser beam focused on a metal
slab.

Several asymptotic expansions are presented
for the small-time response of the tempera-
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